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SHORT METHOD OF ELLIPTIC FUNCTIONS.

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When the last modulus is so small that it may be taken as 0, the denominator, $\sqrt{(1-e_n^2\sin^2\theta_n)}$, evidently reduces to its first term 1; whence $F(0,\theta_n) = \theta_n$; where θ_n denotes the corresponding limit-amplitude. Then,

$$F(e,\theta) = \theta_n \tfrac{1}{2}(1+e^0)\,\tfrac{1}{2}(1+e^{00})\,\tfrac{1}{2}(1+e^{000})\ldots\tfrac{1}{2}.$$

Let

$$A = (1+e^0)(1+e^{00})(1+e^{000})\ldots = \frac{2}{1+b}\cdot\frac{2}{1+b^0}\cdot\frac{2}{1+b^{00}}\ldots = \sqrt{\frac{b^0b^{00}b^{000}\ldots}{b}}.$$

Also let n denote the number of approximations, or of factors in the numerator; then

$$F(e,\theta) = A.(\theta_n \div 2^n) = Ax.$$

When $\theta = \frac{1}{2}\pi$, the formula of amplitude gives $\theta' = \pi$; $\theta'' = 2\pi$; $\theta''' = 4\pi$; etc. Hence, for the quadrantal value,

$$F(e,\tfrac{1}{2}\pi) = A.\tfrac{1}{2}\pi.$$

To aid in judging of the approximation, we take $e=\sin 22\frac{1}{2}^\circ$, or $b=\cos 22\frac{1}{2}^\circ$ then

$$\begin{aligned} e^0 &= \sin 2^\circ16'03'' = .03956, & b^0 &= \cos 2^\circ16'03'' = .99922,\\ e^{00} &= \sin 0^\circ13'28'' = .00039, & b^{00} &= \cos 0^\circ13'28'' = .99999991,\\ e^{000} &= .000\,000\,038\,35+ & b^{000} &= 999\,999\,999\,999\,999\,26. \end{aligned}$$

Again, let $e = \sin 45^\circ$, or $e^2 = \frac{1}{2} = 0.5$; then in common logarithms,

Log e, $\sin 45^\circ$	$= 9.84948500$	Log b, $\cos 45^\circ$	$= 9.84948500$
" e^0, $\sin 9^\circ52'45''$	$= 9.23444863$	" b^0 $\sin 80^\circ07'15''$	$= 9.99351181$
" e^{00}, $\sin 0^\circ25'40'$	$= 7.87330122$	" b^{00} $\sin 89^\circ34'20''$	$= 9.99998788$
" e^{000}, $\sin 0^\circ0'2''\,877$	$= 5.14455457$	" b^{000} $\sin 89^\circ59'57''.1$	$= 9.99999999$
$\frac{1}{4}(e^{000})^2$	$= .000\,000\,000\,048\,65$	" A	$= 0.072\,007\,344\,8.$

$$\theta^{000} = 2\theta^{00} - \left(\frac{1-b^{00}}{1+b^{00}}\right)\sin 2\theta^{00} + \tfrac{1}{2}\left(\frac{1-b^{00}}{1+b^{00}}\right)^2\sin 4\theta^{00} - \ldots.$$

This last series is derived from the formula of amplitudes by the well known trigonometric development. *When θ is less than* $45°$, the last written term can evidently be omitted, and the result will still be exact to ten decimal places; that is $(1-b^{00})\div(1+b^{00}) = \frac{1}{4}e^{00\,2} = \frac{1}{4}\tan^4\frac{1}{2}\varphi^0$. And by reduction, $A = \left(\frac{2}{1+b}\right)^{\frac{1}{4}} \cdot \frac{2}{1+\sqrt{b}} = \frac{2}{\sqrt{[(\cos\varphi+1).\sqrt{(\cos\frac{1}{2}\varphi)}]}}$,

$$F(e,\theta) = A.\tfrac{1}{4}(\theta^{00}-\tfrac{1}{8}e^{00\,2}\sin 2\theta^{00}) = A\tfrac{1}{4}(\theta^{00}-\tfrac{1}{8}\tan^4\tfrac{1}{2}\varphi^0\sin 4\theta^0).$$

When e^2 is greater than $\frac{1}{2}$, we may use the formulas of ascending moduli, Sect. IV, and the third approximate value or θ''' will still be exact to ten decimals.

NOTE 1. Another new scale of amplitudes will be developed in Section VII.

NOTE 2. By shorter process from Sect. VIII, let $\Psi = \theta+2u+v$; $\tan u = \sqrt{b}.\tan\theta$; $\tan v = b.\tan\theta$; then

$$\frac{d\theta}{\sqrt{(1+e^2\sin^2\theta)}} = \frac{1}{1+b}\cdot\frac{1}{1+b^0}\cdot\frac{d\Psi}{\sqrt{(1-e^{00\,2}\sin^2\Psi)}}.$$

NOTE 3. Again, as derived from Jacobi, let the common logarithm of q, that is, $\text{Log}\, q = -0.4342944819 \times \pi A' \div A$; where $A' \div A = F(b, \frac{1}{2}\pi) \div F(e, \frac{1}{2}\pi)$; also $\text{Log}\, 0.4342944822\pi = 0.1349341840$; and by Sect. VII; I

$$\Delta = \sqrt{b}.\frac{1-q-q^4+2q^9-q^{16}-\ldots}{1+q-q^4-2q^9-q^{16}+\ldots};\ \tan\frac{\Psi-\theta}{2} = \Delta.\tan\theta;\ e^0 = \frac{e^3}{(1+\Delta)^4};$$

then
$$\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{1+2\Delta}\cdot\frac{d\Psi}{\sqrt{(1-e^{0\,2}\sin^2\Psi)}}.$$

V. INTEGRATION BY SERIES. As shown by analytical trigonometry, if $e=\sin\varphi$, then dF or

$$\frac{d\theta}{\Delta} = \frac{d\theta}{\sqrt{(1-\frac{1}{2}e^2+\frac{1}{2}e^2\cos 2\theta)}} = \frac{d\theta}{\cos^2\frac{1}{2}\varphi\sqrt{(1+\tan^2\frac{1}{2}\varphi\varepsilon^{2\theta\sqrt{-1}})(1+\tan^2\frac{1}{2}\varphi\varepsilon^{-2\theta\sqrt{-1}})}}.$$

Expanding and integrating

$$\begin{aligned} F = \frac{1}{\cos^2\frac{1}{2}\varphi}\Bigg\{ &\theta\left[1+\left(\frac{1}{2}\right)^2\tan^4\frac{\varphi}{2}+\left(\frac{3}{8}\right)^2\tan^8\frac{\varphi}{2}+\left(\frac{5}{16}\right)^2\tan^{12}\frac{\varphi}{2}+\ldots\right] \\ &-\sin 2\theta\left[\frac{1}{2}\tan^2\frac{\varphi}{2}+\frac{3}{16}\tan^6\frac{\varphi}{2}+\frac{15}{128}\tan^{10}\frac{\varphi}{2}+\ldots\right] \\ &+\frac{\sin 4\theta}{2}\left[\frac{3}{8}\tan^4\frac{\varphi}{2}+\frac{5}{32}\tan^8\frac{\varphi}{2}+\ldots\right] \\ &-\frac{\sin 6\theta}{3}\left[\frac{5}{16}\tan^6\frac{\varphi}{2}+\ldots\right]+\text{etc.}\Bigg\}. \end{aligned}$$

In like manner, for the function E, of the second species,

$$dE = \cos^2\tfrac{1}{2}\varphi\sqrt{[(1+\tan^2\tfrac{1}{2}\varphi.\varepsilon^{2\theta\sqrt{-1}})(1+\tan^2\tfrac{1}{2}\varphi.\varepsilon^{-2\theta\sqrt{-1}})]}d\theta.$$

Expanding and integrating as before,

$$E = \cos^2\tfrac{1}{2}\varphi \left\{ \theta[1+(\tfrac{1}{2})^2\tan^4\tfrac{1}{2}\varphi+(\tfrac{1}{8})^2\tan^8\tfrac{1}{2}\varphi+(\tfrac{1}{16})^2\tan^{12}\tfrac{1}{2}\varphi+\ldots.] \right.$$
$$+ \sin 2\theta[\tfrac{1}{2}\tan^2\tfrac{1}{2}\varphi+\tfrac{1}{16}\tan^6\tfrac{1}{2}\varphi+\tfrac{1}{128}\tan^{10}\tfrac{1}{2}\varphi+\ldots.]$$
$$-\tfrac{1}{2}\sin 4\theta[\tfrac{1}{8}\tan^4\tfrac{1}{2}\varphi+\tfrac{1}{32}\tan^8\tfrac{1}{2}\varphi+\ldots.]$$
$$\left. +\tfrac{1}{3}\sin 6\theta[\tfrac{1}{16}\tan^6\tfrac{1}{2}\varphi+\ldots.]+\text{etc.} \right\}.$$

When $\theta=\frac{1}{2}\pi$ the preceding series for F and E reduce to their first terms. Having computed these, the remaining terms can be accurately derived from each other by the following process from Legendre, (*Fonctions Elliptiques,*) Vol. 1, p. 276.

$$F = \int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = A\theta - A_1\sin 2\theta + A_2\sin 4\theta - A_3\sin 6\theta + \ldots$$

Differentiating once, and again,

$$\frac{dF}{d\theta} = \frac{1}{\sqrt{(1-e^2\sin^2\theta)}} = A - 2A_1\cos 2\theta + 4A_2\cos 4\theta - 6A_3\cos^6\theta + \ldots$$

$$\frac{e^2\sin 2\theta}{2\sqrt{(1-e^2\sin^2\theta)^3}} = 4A_1\sin 2\theta - 16A_2\sin 4\theta + 36A_3\sin 6\theta - \ldots.$$

Multiplying the last equation by $2(1-e^2\sin^2\theta) \div e^2$, or by $(2\div e^2)-1+\cos 2\theta$, and the equation preceding by $\sin 2\theta$, to render the left hand members identical, we equate the coefficients of $\sin 2\theta$, $\sin 4\theta$, ... in the right hand members; thus, we shall find,

$$\begin{aligned} 2\times 3A_2 &= 4A_1[(2\div e^2)-1]-A, \\ 3\times 5A_3 &= 16A_2[(2\div e^2)-1]-1\times 3A_1, \\ 4\times 7A_4 &= 36A_3[(2\div e^2)-1]-2\times 5A_2, \\ 5\times 9A_5 &= 64A_4[(2\div e^2)-1]-3\times 7A_3, \ldots. \end{aligned}$$

Again assuming for E and differentiating, we have,

$$E = \int d\theta\sqrt{(1-e^2\sin^2\theta)} = B\theta + B_1\sin 2\theta - B_2\sin 4\theta + B_3\sin 6\theta - \ldots$$

$$\frac{dE}{d\theta} = \sqrt{(1-e^2\sin^2\theta)} = B + 2B_1\cos 2\theta - 4B_2\cos 4\theta + 6B_3\cos 6\theta - \ldots$$

$$\frac{\frac{1}{2}e^2\sin 2\theta}{\sqrt{(1-e^2\sin^2\theta)}} = 4B_1\sin 2\theta - 16B_2\sin 4\theta + 36B_3\sin 6\theta - \ldots$$

Multiplying the preceding derivative of F by $\frac{1}{2}e^2\sin 2\theta$, to make the left hand member identical with this last equation; than equating the coefficients of $\sin 2\theta$, $\sin 4\theta$, ... in the right hand members,

$$\begin{aligned} 4B_1 &= \tfrac{1}{2}e^2(A-2A_2), \\ 16B_2 &= \tfrac{1}{2}e^2(A_1-3A_3), \\ 36B_3 &= \tfrac{1}{2}e^2(2A_2-4A_4), \ldots. \\ 4n^2B_n &= \tfrac{1}{2}e^2[(n-1)A_{n-1}-(n+1)A_{n+1}]. \end{aligned}$$

TABLE OF ELLIPTIC QUADRANTS OF THE FIRST AND THE SECOND SPECIES.
ALSO THE COMMON LOGARITHM OF q. Here $e = \sin\varphi$.

φ.	$F(e, \frac{1}{2}\pi)$.	$E(e, \frac{1}{2}\pi)$.	Log q.	φ.	$F(e, \frac{1}{2}\pi)$.	$E(e, \frac{1}{2}\pi)$.	Log q.
0°	1.5707963268	1.5707963268	. . . .	25°	1.6489952185	1.4981149284	8.08971
1	1.5709159581	1.5706767091	5.27966	26	1.6556969263	1.4923687111	8.12498
2	1.5712749524	1.5703179199	5.88178	27	1.6627159585	1.4864268037	8.15901
3	1.5718736105	1.5697201504	6.23408	28	1.6700594263	1.4802926638	8.19190
4	1.5727124350	1.5688837196	6.48411	29	1.6777348841	1.4739698872	8.22374
5°	1.5737921309	1.5678090740	6.67813	30°	1.6857503548	1.4674622093	8.25461
6	1.5751136078	1.5664967878	6.83673	31	1.6941143573	1.4607735062	8.28456
7	1.5766779816	1.5649475630	6.97091	32	1.7028352364	1.4539077961	8.31367
8	1.5784865777	1.5631622295	7.08723	33	1.7119246952	1.4468692407	8.34199
9	1.5805409339	1.5611417454	7.18991	34	1.7213908314	1.4396621471	8.36957
10°	1.5828428043	1.5588871966	7.28185	35°	1.7312451757	1.4322909693	8.39646
11	1.5853941638	1.5563997978	7.36510	36	1.7414992344	1.4247603101	8.42271
12	1.5881972125	1.5530808919	7.44119	37	1.7521652365	1.4170749234	8.44835
13	1.5912543820	1.5507319510	7.51128	38	1.7632561841	1.4092397160	8.47342
14	1.5945683409	1.5475545759	7.57625	39	1.7747859092	1.4012597508	8.49796
15°	1.5981420021	1.5441504969	7.63683	40°	1.7867691349	1.3931402485	8.52199
16	1.6019785301	1.5405215741	7.69359	41	1.7992215441	1.3848865914	8.54555
17	1.6060813494	1.5366697976	7.74699	42	1.8121598537	1.3765043258	8.56867
18	1.6104541538	1.5325972877	7.79743	43	1.8256018981	1.3679991659	8.59137
19	1.6151009161	1.5283062961	7.84524	44	1.8395667211	1.3593769973	8.61368
20°	1.6200258991	1.5237992053	7.89068	45°	1.8540746773	1.3506438810	8.63563
21	1.6252336678	1.5190785300	7.93400	46	1.8691475460	1.3418060581	8.65722
22	1.6307291016	1.5141469175	7.97540	47	1.8848086574	1.3328699541	8.67848
23	1.6365174093	1.5090071479	8.01505	48	1.9010830335	1.3238421845	8.69944
24	1.6426041437	1.5036621354	8.05311	49	1.9179975464	1.3147295603	8.72011

φ.	$F(e, \frac{1}{2}\pi)$.	$E(e, \frac{1}{2}\pi)$.	Log q.	φ.	$F(e, \frac{1}{2}\pi)$.	$E(e, \frac{1}{2}\pi)$.	Log q.
50°	1.9355810960	1.3055390943	8.74052	70°	2.5045500790	1.1183777380	9.11748
51	1.9538648093	1.2962780080	8.76068	71	2.5507314496	1.1096434135	9.13609
52	1.9728822663	1.2869537388	8.78059	72	2.5998197301	1.1010621688	9.15484
53	1.9926697557	1.2775739483	8.80030	73	2.6521380046	1.0926503455	9.17376
54	2.0132665652	1.2681465311	8.81979	74	2.7080676146	1.0844252194	9.19289
55°	2.0347153122	1.2586796248	8.83912	75°	2.7680631454	1.0764051131	9.21228
56	2 0570623228	1.2491816206	8.85826	76	2.8326725829	1.0686095330	9.23196
57	2.0803580667	1.2396611753	8.87726	77	2.9025649407	1.0610593338	9.25202
58	2.1046576585	1.2301272242	8.89611	78	2.9785689512	1.0537769204	9.27250
59	2.1300214384	1.2205889958	8.91484	79	3.0617286120	1.0467864993	9.29351
60°	2.1565156475	1.2110560276	8.93347	80°	3.1533852519	1.0401143957	9.31515
61	2.1842132169	1.2015381841	8.95200	81	3.2553029421	1.0337894624	9.33756
62	2.2131946950	1.1920456766	8.97045	82	3.3698680267	1.0278436197	9.36091
63	2.2435493417	1.1825890849	8.98885	83	3.5004224992	1.0223125882	9.38545
64	2.2753764296	1.1731793827	9.00720	84	3.6518559695	1.0172369183	9.41152
65°	2.3087867982	1.1638279645	9.02553	85°	3.8317419998	1.0126635062	9.43962
66	2.3439047244	1.1545466775	9.04385	86	4.0527581695	1.0086479569	9.47054
67	2.3808701906	1.1453478567	9.06218	87	4.3386539760	1.0052585872	9.50569
68	2.4198416537	1.1362443647	9.08055	88	4.7427172653	1.0025840855	9.54798
69	2.4609994583	1.1272496378	9.09897	89	5.4349098296	1.0007515777	9.60564

NOTE. The 10, arbitrarily added, is to be subtracted from Log q; thus for 45°, Log q is really $\bar{2}.63563$. And for 90° Log q is 0.

The equation for A_1 is easily found by substituting the series for F and for E in the identical equation,

$$dE = \Delta d\theta = \Delta^2 d\theta \div \Delta = (1-\tfrac{1}{2}e^2+\tfrac{1}{2}e^2\cos 2\theta)dF.$$

Whence $B = \left(1-\frac{e^2}{2}\right)A-\frac{e^2}{2}A_1$, or $A_1 = A\left(\frac{2}{e^2}-1\right)-\frac{2}{e^2}B.$

Had we developed directly by the binomial theorem, and then integrated, we should have found, *for quadrants*,

$$F(e, \tfrac{1}{2}\pi) = A\tfrac{1}{2}\pi = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 e^2+\left(\frac{1.3}{2.4}\right)^2 e^4+\left(\frac{1.3.5}{2.4.6}\right)^2 e^6+\ldots\right].$$

$$E(e, \tfrac{1}{2}\pi) = B\tfrac{1}{2}\pi = \frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 \frac{e^2}{1}+\left(\frac{1.3}{2.4}\right)^2 \frac{e^4}{3}+\left(\frac{1.3.5}{2.4.6}\right)^2 \frac{e^6}{2}+\ldots\right].$$

Here $$A = -e^2\frac{d}{de}\left(\frac{B}{e}\right).$$

The exponential developments have just given, with $e = \sin\varphi$,

$$A = \frac{1}{\cos^2\frac{1}{2}\varphi}\left[1+\tfrac{1}{4}\tan^4\tfrac{1}{2}\varphi+(\tfrac{3}{8})^2\tan^8\tfrac{1}{2}\varphi+(\tfrac{5}{16})^2\tan^{12}\tfrac{1}{2}\varphi+\ldots\right],$$

$$B = \cos^2\tfrac{1}{2}\varphi\ \left[1+\tfrac{1}{4}\tan^4\tfrac{1}{2}\varphi+(\tfrac{1}{8})^2\tan^8\tfrac{1}{2}\varphi+(\tfrac{1}{16})^2\tan^{12}\tfrac{1}{2}\varphi+\ldots\right].$$

Lastly, to develope after a single decrease of modulus, let t be determined by the relation $\surd b = \surd(\cos\varphi) = (1-t)\div(1+t)$, or

$$e = \sin\varphi;\ e^0 = \sin\varphi^0 = \tan^2\tfrac{1}{2}\varphi;\ t = \tan\tfrac{1}{2}\varphi^0 = \surd e^{00}.$$

For reference, $F(e, \tfrac{1}{2}\pi) = (1+e^0)F(e^0, \tfrac{1}{2}\pi)$, $1+e^0 = (1+t)^2\div(1+t^2)$
$= (1+t)^2\cos^2\tfrac{1}{2}\varphi^0.$

$$A = (1+t)^2.[1+(\tfrac{1}{2})^2t^4+(\tfrac{3}{8})^2t^8+(\tfrac{5}{16})^2t^{12}+(\tfrac{35}{128})^2t^{16}+(\tfrac{63}{256})^2t^{20}+\ldots].$$

When $e = .9851714 = \sin 80°07'15''$, $\varphi^0 = 45°$, $t = \tan 22\tfrac{1}{2}°$; $\frac{1225}{16384}\times t^{16} = .000\,000\,044$. The first omitted term is .000 000 000 0318; so that the written terms would give the value of A exact to ten places.

The common logarithms of the series of squares of binomial coefficients are,
$\text{Log}\,(\tfrac{1}{2})^2 = \bar{1}.39794\,00087$; $\text{Log}\,(\tfrac{3}{8})^2 = \bar{1}.14806\,25354$;
$\text{Log}\,(\tfrac{5}{16})^2 = \bar{2}.98970\,00434$; $\text{Log}\,(\tfrac{35}{128})^2 = \bar{2}.87371\,61496$;
$\text{Log}\,(\tfrac{63}{256})^2 = \bar{2}.78220\,11684$; $\text{Log}\,(\tfrac{231}{1024})^2 = \bar{2}.70662\,40466$.

VI. Inverse Method of Integration. Referring to the close of Section II, let the longest side of the spherical triangle $c = 90°$. The other two sides a, b, or θ, θ', termed complementary, are then connected by the equations,

$$F(e, \theta)+F(e, \theta') = F(e, \tfrac{1}{2}\pi);\ \text{or}\ A.(x+x') = A.\tfrac{1}{2}\pi;$$
$$0 = \cos\theta\cos\theta'-\sin\theta\sin\theta'\surd(1-e^2).$$

If $\Delta = \surd(1-e^2\sin^2\theta)$, $e\sin\theta = \surd(1-\Delta^2)$, $e\cos\theta = \surd(\Delta^2-1+e^2)$; and

so for θ', Δ'. Multiply the preceding equation by e^2, substituting in terms of Δ, Δ', and reducing, since $\Delta' = \sqrt{(1-e^2\sin^2\theta')}$

$$\Delta\Delta' = \sqrt{(1-e^2)} = b.$$

This constancy of the complementary product leads to the preliminary assumptions, $\Delta = \sqrt{b}.f\cos 2\theta$, $\Delta' = \sqrt{b}.\ f\cos 2\theta'$. Here f must denote such a function of $\cos 2\theta$ that the product $f\cos 2\theta . f\cos 2\theta' = 1$; or one function is equal to the reciprocal of the other. And since $2dF = Ad(2x)$ $= d(2\theta) \div \Delta$, and by the Calculus, $d(2x)$ is $-d\cos 2x \div \sqrt{(1-\cos^2 2x)}$, it follows that $f\cos 2\theta$ is $f'\cos 2x$. Changing then to $2x$, since $2x' = \pi - 2x$, as above indicated, we see that f' must denote such a function that $f'\cos 2x$ is the reciprocal of $f'\cos(\pi - 2x)$. This condition is fulfilled by assuming

$$\frac{\sqrt{(1-e^2\sin^2\theta)}}{\sqrt{b}} = \frac{1+a_1\cos 2x+a_2\cos 4x+a_3\cos 6x+a_4\cos 8x+\ldots}{1-a_1\cos 2x+a_2\cos 4x-a_3\cos 6x+a_4\cos 8x-\ldots}$$

Or adding and subtracting the identical equation $1 = 1$, and dividing the one result by the other; also denoting the quotient by S,

$$S = \frac{\Delta-\sqrt{b}}{\Delta+\sqrt{b}} = \frac{a_1\cos 2x+a_3\cos 6x+a_5\cos 10x+\ldots}{1+a_2\cos 4x+a_4\cos 8x+\ \ldots}$$

Multiplying both sides by the right hand denominator, *differentiating* and dividing by $-dx$; since $d\theta = \Delta A dx$,

$$\frac{\Delta-\sqrt{b}}{\Delta+\sqrt{b}}\left(4a_2\sin 4x+8a_4\sin 8x+\ldots\right)+\frac{2e^2\sin\theta\cos\theta . A\sqrt{b}}{(\Delta+\sqrt{b})^2}\times$$
$$(1+a_2\cos 4x+a_4\cos 8x+\ldots) = 2a_1\sin 2x+6a_3\sin 6x+\ldots$$

In the last two equations, omitting all terms of the series after a_3, let us substitute the correlative values, firstly $x = 0$, $\theta = 0$, or $\Delta = 1$. The former equation gives, since, by Section V, $\sqrt{b} = (1-t)\div(1+t)$,

$$a_1 - ta_2 + a_3 - ta_4 = t.$$

But the derivative or latter equation vanishes. Secondly, the correlative values $x = 45°$ or $2x = 90°$, $\Delta = \Delta' = \sqrt{b}$, cause the former equation to vanish, while the latter becomes, since $e\sin\theta = \sqrt{(1-\Delta^2)}$, etc.,

$$a_1+\tfrac{1}{4}A(1-b).a_2-3a_3-\tfrac{1}{4}A(1-b)a_4 = +\tfrac{1}{4}A(1-b).$$

Approximately, omitting a_3, a_4, and taking A, b in terms of t from Section VI, we find from the last two equations,

$$a_1 = t + \tfrac{1}{8}t^5 + \ldots,\ a_2 = \tfrac{1}{8}t^4 + \ldots$$

Again *differentiating*, and substituting in place of x and θ or Δ, as before, we find a third equation; from which and the two former, omitting a_4, we find by elimination, exact to another power of t,

$$a_1 = t + \tfrac{1}{8}t^5 + \tfrac{15}{256}t^9 + \ldots,\ a_2 = \tfrac{1}{8}t^4 + \tfrac{1}{16}t^8 + \ldots,\ a_3 = \tfrac{1}{256}t^9 + \ldots$$

By repeated differentiations, other equations can be found in the same way, and the approximation carried to any extent, disclosing this remark-

able law; let $t = (1-\surd b)\div(1+\surd b)$; $q = \frac{1}{2}t + \frac{1}{16}t^5 + \frac{15}{512}t^9 + \frac{150}{8192}t^{13} + \frac{1707}{131072}t^{17} + \ldots$; then $a_1 = 2q$, $a_2 = 2q^4$, $a_3 = 2q^9$, $a_4 = 2q^{16}$, $a_5 = 2q^{25}$, $a_6 = 2q^{36}$, ... This has been fully demonstrated in a different way by its illustrious discoverer, Jacobi. He has given in the twenty-sixth volume of Crelle's Journal an extended table of the values of q for every $6'$ of φ. In this, and in other ways may be demonstrated one of the most important results, the inverse relation,

$$\frac{\surd(1-e^2\sin^2\theta)}{\surd b} = \frac{1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\ldots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots}$$

In the next place, for determining the integral x through $\cos 2x$, by *reversion of series*, we take the second form, found before,

$$S = \frac{\Delta-\surd b}{\Delta+\surd b} = \frac{2q\cos 2x + 2q^9\cos 6x + \ldots.}{1+2q^4\cos 4x+2q^{16}\cos 8x+\ldots}$$

Multiplying both sides by the right hand denominator, and changing $\cos 4x$, $\cos 6x$, etc., to powers of $\cos 2x$, also omitting the sixteenth and higher powers of q.

$$S(1-2q^4+4q^4\cos^2 2x) = (2q-6q^9)\cos 2x(1+4q^8\cos^2 2x).$$

Dividing both members by the right hand factor, omitting q^{12},

$$S[1-2q^4+4(q^4-q^8)\cos^2 2x] = (2q-6q^9)\cos 2x.$$

This equation, being a common quadratic, is easily resolved through an auxiliary arc v; the well known form of the result will be, when P and Q are functions of q only, $\sin v = PS$; $\cos 2x = Q\tan\frac{1}{2}v$.

To determine P and Q independently of q, we first elimiminate v, since $\sin v = 2\tan\frac{1}{2}v \div (1+\tan^2\frac{1}{2}v)$, and in the first and third derivatives of the result, we make $x = 45^\circ$, and $\Delta = \surd b$, which gives, if $\sin\varphi = e$, $\cos\psi = b$; $t = (1-\surd b)\div(1+\surd b)$, the practical solution,

$$Q = \sqrt{\left[\frac{48}{A^2(1+6b+b^2)-8}\right]} = \frac{2}{t^2} - \frac{9}{16}t^2 - \frac{165}{1024}t^6 - \ldots$$

$$\sin v = \frac{8}{(1-b)AQ}\cdot\frac{\Delta-\surd b}{\Delta+\surd b};\ \cos 2x = Q\tan\tfrac{1}{2}v.$$

The resulting integral x should be exact to q^{12}, that is to ten decimal places when φ is less than 70°, or exact to seven decimals when φ is less than 83°. The auxiliary S varies from the limit $-t$ to $+t$; so that v may be positive, 0, or negative; and $2x$ greater or less than 90° or $\frac{1}{2}\pi$. When φ is near to 90°, the value of x may be found from the standard equation by the process of trial and error, to any degree of accuracy. Finally $F = A\times x$.

(*To be concluded in No. 1, Vol. V.*)